Michigan Technological University **Division of Teacher Education**

Official Lesson Plan Form

Lesson Title:___Complex Polynomial Equations________ Date:__________________________

Subject/Class:__Mathematics: Pre-Calculus____________ Grade Level: ____11 / 12_________

MI Framework Standard(s) [(+) indicates an AP standard]:
N-CN:

3. Use complex numbers in polynomial identities and equations.
   a. Solve quadratic equations with real coefficients that have complex solutions.
   b. (+) Extend polynomial identities to the complex numbers. For example, rewrite x^2+4 as (x +2i)(x –2i).
   c. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

F-IF:

1. Analyze functions using different representations
   a. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.
2. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

Objectives:
Upon completion of the lesson, students will be able to...
1. Solve polynomial equations with complex coefficients that have complex solutions.
2. Describe the Fundamental Theorem of Algebra and apply it to polynomial equations to find complex solutions.
3. Show which algebraic identities apply to complex numbers and which require tweaking.
4. Apply complex numbers to real world problems.
5. Define common mathematical proof terminology.
6. Construct logical, consistent, effective mathematical proofs.

Rationale for the Lesson:
In this lesson, students will apply the understanding of complex numbers they have built over the previous lessons to polynomial equations with complex coefficients. The Fundamental Theorem of Algebra will be discussed and used to find polynomial roots. Through this lesson, students will gain a deeper understanding of the previously fully-real world of polynomial equations and ideally will grasp a more ordered set of rules. Additionally, the final project for the unit will be introduced and students will start independent work on producing a proof for a complex hypothesis of their choice.

Student Prior Knowledge/Common Misconceptions:
Students should have a fairly solid knowledge of working problems with complex numbers involved and converting between various forms (Cartesian and Polar at this point) from the previous lessons. Somewhat advanced manipulations of polynomials and related identities will be helpful as well, though as long as the basic understanding of polynomial equations is present, it'll be fine. Misconceptions may include dismissing imaginary solutions inadvertently through things such as z^7=128 leading to just z=2 rather than recognizing there must be seven solutions.

Materials and resources needed for lesson:

- Whiteboard, markers
- Worksheets / example problems (self-made if time allows)
- Paper/pencil
- Helpful: computer with graphing program installed such as MATLAB, Mathematica

References, when appropriate:

Opening

Time	Procedures/Details
5m 5m [10m total]	Warm up problem: Find the roots of a quadratic equation (in this case one real and one complex root). Use this to lead into a question: why are its roots one real and one complex? What are the rules? Any questions from the homework?

Instructional Activities

Time	Procedures/Details
15m	Ask students to define a full set of rules for when an equation can have X many roots, and how many of the roots can be real or complex. This can be relatively unstructured; shouldn't take too long. Remember: complex coefficients = at least one unpaired complex root (possibly more), real coefficients = only real roots *or* all complex roots are conjugate pairs. (See Notes for examples). Emphasize the "starting with solutions, finding patterns" idea and authentic procedures used by mathematicians.
20m	Introduce the final project and form teams (2-3 students per team, I think. Not too many). Provide possible theorems to prove and allow groups to either start planning or start coming up with a different theorem to prove. Allow time to play around with numbers, help groups figure out what their plan is. Every group should have a concrete plan before the end of the day. Assign reading (linked in notes section).
5m	Regroup, confirm topics and presentation order. End of first day.
20m	Next day: Review proof notation and conventions from the reading. Clear up confusion especially on notation. Likely a table or something visual: for all, there exists, such that, implies, and/or/not/if then/iff, etc. See notes for additional questioning strategies.
10m	Go over the other proof reading (Tips for writing proofs), have class debate whether the ten tips and the "good" proof are actually good and how to improve it.
20m [90m total]	Remainder of class: Small-ish groups, provide simple complex theorems similar to the ones we've already done (ie operations on complex numbers, etc.). Give maybe 5-10 minutes per problem to work on these and regroup to share out at the end. End of second day.

Closing

Time	Procedures

50m [50m total]	Allow one class period to work on projects.

Assessment (including diagnostic, formative and/or summative):
(Diagnostic/formative)
Questions / observations during class:

- During the unstructured discussion just keep an eye out for students who aren't contributing and ask their thoughts on the matter.
- Everyone's topic should be at least a minimum complexity; if it's too short, try suggesting other related problems they can work on in addition rather than just changing entirely.
- "Why is your topic interesting? Do you have an idea on how to go about proving it? Presenting this to the class, who won't have the same background as you?"

(Summative)
Proof Project:
Rubric and instructions attached, but the general gist is to find an interesting or neat pattern involving complex numbers (possible choices are provided for those who don't want to come up with one) and explain why it happens, as well as prove that it holds under certain conditions (or all conditions).

Additional Teacher Notes:

Example equations for students to work with to find patterns:

$$x^2 + 1 = 0$$
$$x^2 + (1 + i)x + i = 0$$
$$x^2 - 1 = 0$$
$$x^3 - x = 0$$
$$x^3 - ix^2 - x + i = 0$$
$$x^3 - x^2 + x - 1 = 0$$

These six equations all have roots of $\pm 1, \pm i, or\ 0$. They should be fairly straightforward to build on the conjugate knowledge and Fundamental Theorem of Algebra understanding students should have.
Ask students to fill out the following table on the board or similar.

	Real coefficients	Complex coefficients
Only real roots	No problems	Impossible
Real and complex roots	Complex roots must be conjugate pairs	Complex roots must leave at least one unpaired
Only complex roots	Roots must be conjugate pairs	Must be at least one unpaired root

Readings:
https://math.berkeley.edu/~hutching/teach/proofs.pdf . Notation and proof structure. Instructions: read through page 9, skim all of section 3 and section 4.1.

http://www.ms.uky.edu/~kott/proof_help.pdf . Tips for writing a good proof. Instructions: read the entire document, determine which tips you agree or disagree with.

Questioning notes:
We're looking for two things here: the basics of proof notation and the basics of proof theory. For notation, here's a table that students should be able to fill out on their own:

Phrase	Symbol	Meaning
For all	∀	For every A, something is true.
There exists	∃	Some A exists with some property.
And	and	If A and B are true, then C is. If either A or B is false, then C is.
Or	or	If A or B is true, then C is. If A and B are false, then C is.
Such that	s.t.	A has some property, detailed next.
If and only if	iff, ⇔	If A is true, B is true. If A is false, B is false. A and B are equivalent.
Implies	⇒	If A is true, then B is true. (if A is false, $A \Rightarrow B$ is vacuously true.
Element of	∈	A is an element of B (set-wise, range, or otherwise)

Ask students to come up with some example for each element of notation where it is used and correct.

For theory, much of this will be a review of pages 1-9 in the reading. Possible questions / directions to go in:

- What is a proof? What is its objective?
- What makes a proof "good?" What makes a proof "bad?"
- What is a statement?
- What is a logical operator? Give an example.
- What is a quantifier? Give an example.
- What does vacuously true mean?
- When is it appropriate to use the "it is obvious that…" technique to prove something?
- (Walk through the proof dialogue; any questions about or problems with that?)
- (Walk through an example of proof by cases, proof by contradiction, and proof by induction)

Possible theorems for proof project:

- De Moivre's Theorem: z^n = r^n (cos(n*theta) + isin(n*theta)) – this is an extension of a concept we proved in class for n = 2.
- If no one has solved the challenge problem C1) from the first homework: Prove that if z_0 is a root of the polynomial equation $z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n = 0$, with $z = a + bi$ and $a_n \in \mathbb{R}$ (meaning that z is complex and all coefficients a_n are real), then $\overline{z_0}$ (the complex conjugate of z_0) is also a root of the equation.
- Roots of unity: show that the solution to z^n=1 is cos(2k*pi/n)+isin(2k*pi/n) for k = 0, 1, 2, …, n-1

- Conjugate madness: determine the result of $\overline{z+\overline{z+\overline{z+\overline{z+\cdots}}}}$ for z = a+bi. What about $\overline{z_0+\overline{z_1+\overline{z_2+\overline{z_3+\cdots}}}}$ for $z_0 = a_0+b_0i$, $z_1 = a_1+b_1i$, …?
- Prove the Fundamental Theorem of Algebra using Rouché’s Theorem (see HW 3).
- Additional theorems may be added as time permits.