Name Key ________________________________ Date ________________

Directions: Find all of the roots of each polynomial.

1) $x^2 + 16 = 0$

$$x^2 = -16$$
$$x = \pm 4i$$

2) $x^2 + 6x + 18 = 0$

$$(x+3)^2 + 9$$
$$(x+3)^2 = -9$$
$$x = -3 \pm 3i$$

3) $x^4 = 16$

Roots of unity. $x = \pm 2, \pm 2i$
Can draw a graph to figure this out, or use the formula:

$$2\left(\cos\left(\frac{2\pi k}{4}\right) + i\sin\left(\frac{2\pi k}{4}\right)\right), k = 0,1,2,3$$

4) $x^2 - 4x - 12 = 0$

$$(x-6)(x+2)$$
$$x = 6, -2$$

(This is somewhat of a trick question – no complex numbers required)

5) $x^2 + 6x + 13 = 0$

$$x^2 + 6x + 9 = -13 + 9$$
$$(x+3)^2 = -4$$
$$x = -3 \pm 2i$$

6) $x^3 - 3x^2 + 4x - 12 = 0$

$$x^2(x-3) + 4(x-3) = 0$$
$$(x-3)(x^2+4) = 0$$
$$x = 3, \pm 2i$$

Challenge Problems
Directions: These are optional bonus problems you may attempt if you desire.

C1) Rouché's Theorem states that for any two functions f and g that have no singularities in a certain region, if $|f| > |g|$ for all values on the boundary of the region, then f and g have the same number of zeroes in the region.
The proof of this theorem usually involves taking the inequality $|(f+g) - f| < |f - 0|$ (true by |f| > |g|) and noting that the Euclidean distance between f+g and f must go to 0 when f=0, so f+g always has a zero when f has a zero, so g must also have a zero when f does.
When combined with a few more tools from complex analysis we don't yet have, this theorem provides a straightforward way to prove the Fundamental Theorem of Algebra.

Using Rouché's Theorem, can you prove that all the zeroes of $x^4 + 6x + 3$ have magnitude less than 2?
Hint: Consider $f = x^4$, which we know has four zeroes (all at x=0) of magnitude less than 2.

Let $f = x^4$ and $g = 6x + 3$.
On a circle of radius 2, $|x^4| = 16$.
Similarly, on that circle, $|6x + 3| = 15$.
Thus the hypotheses hold and f, g have the same # of zeroes in this region (four).
Then f+g has all four of its zeroes in the same region as well by function addition.

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