Lesson Title:___The Complex Number System_____   Date:_____ _____

Subject/Class:__Mathematics: Pre-Calculus_____   Grade Level:_____11 / 12_____

MI Framework Standard(s) [(+) indicates an AP-level standard]:
N-CN:
1. Perform arithmetic with complex numbers
    a. Know there is a complex number i such that $i^2 = -1$, and every complex number has the form a + bi with a and b real.
    b. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
    c. (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Objectives:
Upon completion of the lesson, students will be able to...
1. Define the complex unit *i* and how it relates to the real numbers.
2. Perform arithmetic with complex numbers.
3. Extend the real number system to the complex numbers - identify which properties still hold and which require tweaking.
4. Find the conjugate of a complex number and use it to help solve equations and convert forms.
5. Manipulate expressions containing *i* to convert to and from rectangular form.

Rationale for the Lesson:
In this lesson, students will be introduced to the basics of arithmetic with complex numbers. Having at least a basic understanding of *i* and the complex number system is helpful in many kinds of problems likely to be encountered in later math courses (ie finding roots of polynomials, graphical representations of complex phenomena, shortcuts to simplify algebra, etc). Problem solving with complex numbers is important for later calculus and advanced college mathematics, as well as in engineering fields (particularly physics, electromagnetics, photonics, and electrical engineering).

Student Prior Knowledge/Common Misconceptions:
Students will need a fairly decent understanding of algebra and number-juggling, including factoring and completing the square. Previous experience with number systems (ie standards N-RN 1, 2, 3) is a bonus but not necessary; be prepared to review them in detail.
Algebra skills are very important for manipulating complex numbers and *i* is often dropped or forgotten about - watch out, be careful. Placing a letter in front of a coefficient can throw students - work on understanding letters can go wherever (Ahmad, 2014).
Work on ability to apply knowledge to novel problems, rather than just working examples and changing them slightly for the test. Students often have trouble thinking creatively and applying their new skills (Selden, 1994).

Materials and resources needed for lesson:
• Whiteboard, markers
• Worksheets / example problems (self-made if time allows)

- Paper/pencil

References, when appropriate:

Ahmad, A. & Shahrill, M. (2014). Improving Post-Secondary Students' Algebraic Skills in the Learning of Complex Numbers. International Journal of Science and Research, 3(8).

Selden, J., Selden, A., & Mason, A. (1994). Even good calculus students can't solve nonroutine problems. Research Issues in Undergraduate Mathematics Learning. 19-26.

Opening

Time	Procedures/Details
5m	Write simple problem on board related to previous unit on quadratics (say, $x^2+4=0$). Ask students to solve using previously known methods, go over as a class (subtract, square root). Discuss why this doesn't work due to previous material (sqrt of a negative number), ask why it can't work.
5m	Use properties of square roots to separate out the problematic part of previous eqn (-4 -> -1 * 4), then simplify to ±2sqrt(-1) and call that "i" to solve as = ±2i. Explain that i is the imaginary unit and has some interesting properties. Reconnect to extending the real number system.
[10m total]	

Instructional Activities

Time	Procedures/Details
10m	Determine as a class which operations are closed under the real and complex systems, including +, -, *, /, sqrt, exp. This may take a fair while if we need to review number systems in a fair amount of detail.
(30m total)	Define a complex number in Cartesian form as a+bi, where a, b are real numbers. Work through each normal operation (+, -, *, /) and find simple formulae for each. Use a+bi (operation) c+di, work as full class to figure it out on the board.
5m	+: straightforward. a+bi + c+di = (a+c) +(b+d)i
5m	-: straightforward. a+bi – c+di = (a-c) + (b-d)i
10m	*: more complicated. (a+bi)(c+di) -> FOIL -> (ac-bd)+(ad+bc)i. This allows us to introduce powers of i. Break the unit down into sqrt(-1) and have students determine the pattern: i, -1, -i, 1, repeat.
10m	Go over example problems and basic arithmetic with complex numbers. End of first day.
5m	Pick up from yesterday; warm up problem for class using skills that should have been built yesterday.
20m	/: most complicated. $(a+bi)/(c+di) = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right) i$. This allows us to introduce conjugate pairs: a+bi and a-bi. Introduce these to get rid of the c+di on the bottom and connect it to Pascal's triangle and the idea of even powers of i being real to understand why $(a+b)(a-b)=(a^2-b^2)$ allows us to get rid of i when it's somewhere we don't want.
10m	Go a little farther into conjugate notation (z-bar) and give conjugate identities: (z_1 (op) z_2)-bar = (z_1)-bar (op) (z_2)-bar. Explain why this is the case with general terms (a+bi, a-bi).
10m	Review more advanced examples, including a refresher on completing the square, but with imaginary numbers. (This will help with #2, #3 on the homework; just take those problems and change the numbers a little)
[85m total]	

Closing

Time	Procedures
5-ish m (whatever time is left in class)	Review the operations we've gone over so far, graphically. a+bi as a number in the complex plane, algebraic manipulation converting to graphical numbers, see what patterns emerge. Connects to next time. Assign homework, hand out worksheet. Allow time to work if there's any class time left.

Assessment (including diagnostic, formative and/or summative):

Questions / observations during class:
- During opening, how students are solving the equation and how far they're willing to simplify. See if the concepts from last time have stuck.
- Why does sqrt(-a) not work? Their explanation here can help bring out misconceptions, since there's really no reason it shouldn't work. (Good answer: something like a^2 is always positive, so sqrt(-a) isn't a thing. Bad answer: Not defined for negative #'s. (okay, but *why*?)
- Beginning of main block: what is closure? How do I know if an op is closed on a set? What are the real numbers? Do we remember rationals? Integers? Naturals? This will help determine how much students recall about number systems.
- Main block of main block: misconceptions can come up here if students screw up the algebra or drop the *i*. Be on the lookout for errors or weaknesses with algebra; we're going to need high skill for later.
- Figuring out the pattern of *i*, -1, -*i*, 1 shouldn't be too hard; if students are struggling, we're missing something fundamental about the properties of square roots and negative numbers and need to stop and figure that out.
- Conjugate pairs: if students can come up with the idea on their own, they're advanced. Otherwise, prompt it by asking what binomial can turn (a+b) into (a^2-b^2), extend to (a+bi). Use real numbers as a scaffold if needed; (x+2)(x-2)=(x^2-4).
- Ask students to determine a-bar (of a real number). Obviously, a-bar = a. That will be useful on the homework.

Homework: a worksheet on basic number-juggling and equation solving. Nothing too difficult, except for the challenge problems at the end for advanced students. Worksheet will be attached to this plan.

Additional Teacher Notes:

If needed, we can introduce the concept of magnitude as |z| = sqrt(a^2+b^2), but I think it fits better and more naturally with the next section once we get into graphical representation.